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The number of extremal components of a rigid measure ☆

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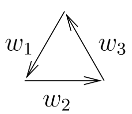
ABSTRACT

The Littlewood–Richardson rule can be expressed in terms of measures, and the fact that the Littlewood–Richardson coefficient is one amounts to a rigidity property of some measure. We show that the number of extremal components of such a rigid measure can be related to easily calculated geometric data. We recover, in particular, a characterization of those extremal measures whose (appropriately defined) duals are extremal as well. This result is instrumental in writing explicit solutions of Schubert intersection problems in the rigid case.

1. Introduction

Our main object of study is a special class of measures in the plane, which we now define. Start with three unit vectors w_1, w_2, w_3 in $\mathbb{R}^2$ such that $w_1 + w_2 + w_3 = 0$.



The measures μ we are interested in are supported in a finite union of lines parallel to one of these three vectors, and satisfy the following two conditions.

(1) On each segment which does not intersect other segments in its support, μ is proportional to length; the constant of proportionality is the *density* of μ on that segment. The density of μ will be considered to be zero on segments outside its support.

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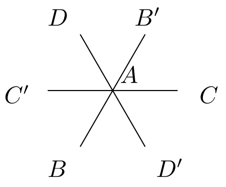
(2) For any point $A \in \mathbb{R}^2$, we have

$$\delta_1^+(\mu, A) - \delta_1^-(\mu, A) = \delta_2^+(\mu, A) - \delta_2^-(\mu, A) = \delta_3^+(\mu, A) - \delta_3^-(\mu, A), \tag{1.1}$$

where $\delta_j^\pm(\mu, A)$ is the density of μ on the segment $\{A \pm tw_j \colon t \in (0, \varepsilon)\}$ for small ε.

Condition (2) is only relevant for the (finitely many) points A for which at least three of the numbers $\delta_j^\pm$ are different from zero. These are called *branch points* of the measure μ. The following figure represents a branch point A of a measure μ, along with six segments of equal length to which μ may assign positive density.

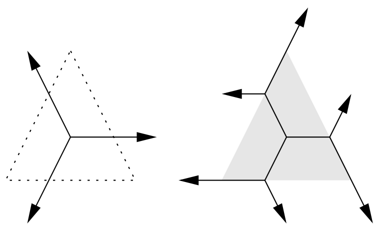


If these segments contain no other branch points, then condition (1.1) amounts to the identity

$$\mu(AB) - \mu\big(AB'\big) = \mu(AC) - \mu\big(AC'\big) = \mu(AD) - \mu\big(AD'\big).$$

We denote by $\mathcal{M}$ the convex cone consisting of all measures satisfying conditions (1) and (2).

Assume now that r is a positive number, and denote by Δ_r the (closed) triangle with vertices 0, rw_1, and $r(w_1 + w_2)$. The cone $\mathcal{M}_r \subset \mathcal{M}$ consists of those measures μ whose branch points are contained in Δ_r, and whose support outside Δ_r consists of a finite number of half-lines of the form $\{A + tw_j \colon t > 0\}$ with $A \in \partial\Delta_r$ and $j \in \{1, 2, 3\}$. Analogously, $\mathcal{M}_r^*$ consists of those measures μ whose branch points are contained in Δ_r, and whose support outside Δ_r is contained in a finite number of half-lines of the form $\{A - tw_j \colon t > 0\}$ with $A \in \partial\Delta_r$. A point $A \in \partial\Delta_r$ such that $\{A \pm tw_j \colon t > 0\}$ is contained in $\operatorname{supp}(\mu) \setminus \Delta_r$ is called an *exit point* of μ, and the corresponding density an *exit density*. The following figure shows the supports of a measure in $\mathcal{M}_r$ and of a measure in $\mathcal{M}_r^*$. In the case of $\mathcal{M}_r$, the boundary of Δ_r is indicated by a dotted line, while for $\mathcal{M}_r^*$ the triangle is colored light gray. The arrows indicate the half-lines in the support.



Note that the first measure has 3 exit points, while the second one has 6. The exit points and exit densities are always determined by the restriction of μ to Δ_r.

It will be useful to distinguish a subset of the exit points, called the *attachment points* of μ (or of the support of μ). An exit point of μ will be called an attachment point if either

(a) it is not a corner of Δ_r, or
(b) it is a corner of Δ_r, and the half-line through that point, parallel to the opposite side of Δ_r, is in the support of μ.

The two supports pictured above have three attachment points each.

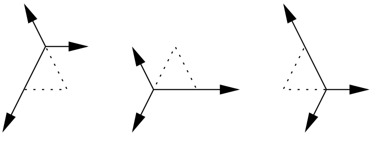
A measure $\mu \in \mathcal{M}_r$ is said to be *rigid* if there is no other measure $\nu \in \mathcal{M}_r$ with the same exit points and same exit densities as μ. An analogous definition applies to $\mathcal{M}_r^*$. A measure $\mu \in \mathcal{M}$, $\mu \neq 0$, is said to be *extremal* if any measure $\nu \in \mathcal{M}$ satisfying $\nu \leqslant \mu$ is of the form $c\mu$ for some constant c. It was shown in [1] (and the result will be reviewed below) that rigid measures in $\mathcal{M}_r$ can be written in a unique way as sums of extremal measures with distinct supports.

There is a duality which associates to each nonzero measure $\mu \in \mathcal{M}_r$ a measure $\mu^* \in \mathcal{M}^*_\omega$, where $\omega = \omega(\mu) > 0$ is the weight of μ, defined in Section 2 below. If μ is rigid, then μ^* is rigid as well. We will denote by $\mathrm{ext}(\mu)$ and $\mathrm{ext}(\mu^*)$ the number of extremal summands of a rigid measure μ and the corresponding number for μ^*. The number of attachment points of μ will be denoted $\mathrm{att}(\mu)$. Our main result is as follows.

Theorem 1.1. *For every rigid measure $\mu \in \mathcal{M}_r$, we have*

$$\mathrm{ext}(\mu) + \mathrm{ext}\big(\mu^*\big) = \mathrm{att}(\mu) + 1.$$

In particular, if $\mathrm{ext}(\mu) = \mathrm{ext}(\mu^*) = 1$, we deduce that $\mathrm{att}(\mu) = 1$, in which case the support of μ must be in one of the positions pictured below.



Each of these measures has three exit points, but only one of them is an attachment point. The attachment point is the same as the unique branch point of the measure, and it must be deemed as *two* exit points. This is precisely [1, Proposition 5.2].

The remainder of this paper is organized as follows. In Section 2 we describe some basic properties of measures, as well as the duality $\mu \mapsto \mu^*$. This material is contained, more or less explicitly, in [4] and [5]. Section 3 begins with the mechanics of decomposing rigid measures into their extremal summands, and a linear algebraic consequence of this decomposition. Theorem 3.2 shows how our main result reduces to calculating the dimension of a certain convex set. Its proof requires the study of measures obtained by immersing a tree (Section 4) and of the small perturbations of such immersions (Section 5). We conclude the paper with an illustration of the main result.

The role of measures in the study of the Littlewood–Richardson rule and in the intersection theory of Grassmannians was first pointed out in [5]. The extremal structure of rigid measures was described in [1], where it was shown how the associated intersections of Schubert varieties can be written explicitly. We recall briefly how this result is used in solving intersection problems. Given an arbitrary nonzero measure μ, we can always write it as a sum $\mu = \mu_1 + \mu_2 + \cdots + \mu_k$ of nonzero inequivalent extremal measures. If μ is rigid, the summands $\mu_1, \ldots, \mu_k$ are uniquely determined (up to permutations). If, in addition, μ has only integer densities, then so do these extremal summands. Assuming that explicit formulas are known for the solution of the intersection problem associated to each summand μ_j, it is shown in [1] that such a formula can be found for the intersection problem associated to μ. The intersection problem associated to a rigid extremal measure can be further reduced by passing to the dual measure μ^* which is no longer extremal in general. Repeated application of these procedures solves all the intersection problems for which the measure μ is rigid.

Rigid extremal measures have an underlying tree structure which was described in [1,2], and which also plays a role in this paper. We should observe that for integer values of r, the set $\mathcal{M}_r$ used in this paper is not the same as its namesake in [1] and [2]. Indeed, in those papers the branch points are always of the form $p_1 w_1 + p_2 w_2$ with integer p_1 and p_2. This hypothesis is natural when dealing with intersection problems, but the arguments of [1,2] do not depend on it in an essential manner.

2. Weight, trace identity, and dual

Consider a measure $\mu \in \mathcal{M}_r$ for some $r > 0$. We begin by establishing two identities. The first one (2.1) allows us to define the weight of μ, while (2.2) is the trace identity which plays an important role in our arguments.

Eq. (1.1) is equivalent to

$$\sum_{j=1}^{3}\sum_{\varepsilon=\pm}\varepsilon\delta_j^{\varepsilon}(\mu,A)w_j=0,$$

and therefore

$$\sum_{A}\sum_{j=1}^{3}\sum_{\varepsilon=\pm}\varepsilon\delta_j^{\varepsilon}(\mu,A)w_j=0,$$

where the sum is extended over all the branch points of μ. Assume that A and B are two branch points such that the segment AB contains no other branch points of μ. If AB is a positive multiple of w_j, and δ is the density of μ on AB, then $\delta_j^{+}(\mu,A)=\delta_j^{-}(\mu,B)=\delta$, so that these two terms will cancel out in the sum above. The only remaining terms correspond therefore to the exit densities of μ. Denote by $\alpha_1^{(j)},\alpha_2^{(j)},\dots,\alpha_{k_j}^{(j)}$ the exit densities in the direction of w_j. We deduce that

$$\sum_{j=1}^{3}\left[\sum_{i=1}^{k_j}\alpha_i^{(j)}\right]w_j=0,$$

which in turn implies

$$\sum_{i=1}^{k_1}\alpha_i^{(1)}=\sum_{i=1}^{k_2}\alpha_i^{(2)}=\sum_{i=1}^{k_3}\alpha_i^{(3)}. \tag{2.1}$$

The common value of these sums is called the *weight* of μ, and will be denoted $\omega(\mu)$.

There is another identity involving exit densities, which we will call the *trace identity* because of its connection with traces of matrices. One way to deduce it is to observe that an arbitrary measure $\mu\in\mathcal{M}$ represents the second differences of a convex function on $\mathbb{R}^2$. More precisely, there exists a (necessarily continuous) convex function $f:\mathbb{R}^2\to\mathbb{R}$ with the following property: for any two equilateral triangles ABC, $A'BC$ whose interiors do not intersect the support of μ, we have

$$f(A)+f\left(A'\right)-f(B)-f(C)=\mu(BC).$$

It suffices, of course, to require this condition for triangles whose sides are parallel to the vectors w_j. Thus, the function f is affine on each connected component in the complement of the support of μ, and the piecewise constant function $df(x+tw_j)/dt$ jumps at the points where the line $x+tw_j$ intersects the support of μ transversally, the amount of each jump being equal to the density at the intersection point. It is easily seen that condition (1.1) ensures that the various affine pieces of f fit together around each branch point of μ. The function f is uniquely determined by its values at three noncollinear points; these values can be prescribed arbitrarily. We will write $\mu=\nabla^2 f$ to indicate this relationship between f and μ. In the terminology of [4] and [3], $-f$ (or its restriction to Δ_r) is a *hive*.

Assume now that $\mu\in\mathcal{M}_r$ has exit densities $\{\alpha_i^{(j)}:\ 1\leqslant i\leqslant k_j\}$ in the direction of w_j, $j=\{1,2,3\}$, and the corresponding exit points are $A_i^{(j)}$. Denote by $X_1=rw_1$, $X_2=r(w_1+w_2)$, and $X_3=0$ the vertices of Δ_r. The points $A_i^{(j)}$ are on the segment X_jX_{j+1}. We will denote by $x_i^{(j)}$ the distance from $A_i^{(j)}$ to X_j. The number $x_i^{(j)}\in[0,r]$ will also be called the *coordinate* of $A_i^{(j)}$. Consider now a convex function f such that $\mu=\nabla^2 f$. The function $df(X_j+tw_{j+1})/dt$ jumps by $\alpha_i^{(j)}$ at $t=x_i^{(j)}$, and therefore

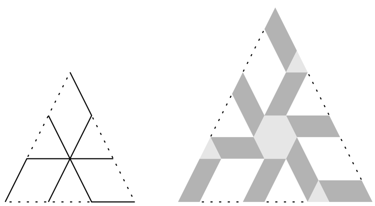
$$f(X_{j+1})-f(X_j)=r\beta_j+\sum_{i=1}^{k_j}\alpha_i^{(j)}\left(r-x_i^{(j)}\right)=r\beta_j+r\omega(\mu)-\sum_{i=1}^{k_j}\alpha_i^{(j)}x_i^{(j)},$$

where $\beta_j = df(X_j + tw_{j+1})/dt$ for $t < 0$. The function f can be chosen so that it is identically zero in the angle bounded by $\{tw_2\colon t \geqslant 0\}$ and $\{tw_3\colon t \geqslant 0\}$. In this case, we have $\beta_3 = 0$ and $\beta_1 = \beta_2 = -\omega(m)$, so that adding the above identities for $j = 1, 2, 3$ yields the *trace identity*

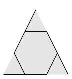
$$\sum_{i=1}^{k_1} \alpha_i^{(1)} x_i^{(1)} + \sum_{i=1}^{k_2} \alpha_i^{(2)} x_i^{(2)} + \sum_{i=1}^{k_3} \alpha_i^{(3)} x_i^{(3)} = r\omega(\mu). \tag{2.2}$$

Let us observe that the exit point $A_i^{(j)}$ is an attachment point of μ precisely when its coordinate $x_i^{(j)}$ is not zero. Thus, only attachment points contribute significantly to the sum in the left-hand side.

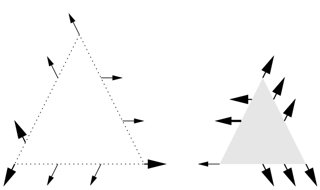
Next we discuss the dual of a measure $\mu \in \mathcal{M}_r$. For this purpose, we construct the *puzzle* of $\mu|\Delta_r$ as follows. We translate the connected components of the complement $\Delta_r \setminus \operatorname{supp}(\mu)$ away from each other in such a way that the parallelogram formed by the two translates of a side AB in the support of μ has two sides which are 60° clockwise from AB, and have length equal to the density of μ on AB. Condition (1.1) ensures that these pieces fit together. There is a polygon, corresponding to each branch point, which is not covered by these pieces, and which has sides equal to the densities of the segments meeting at that point. This process is illustrated below, where the white areas are the translated components of $\Delta_r \setminus \operatorname{supp}(\mu)$, the connecting parallelograms are dark gray, and the polygons corresponding to branch points are light gray. The solid lines in the first picture represent the support of μ, and all of them are taken to have the same density for this figure.



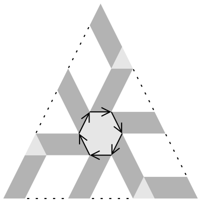
The three kinds of pieces (white, dark gray parallelograms and light gray) form the puzzle of μ, and the process of passing from a measure to its puzzle is called *inflation*. The puzzle pieces cover an equilateral triangle with side $r + \omega(\mu)$ which can, and generally will, be assumed to be precisely $\Delta_{r+\omega(\mu)}$. We can now apply a dual process of **-deflation* to this puzzle as follows. Consider a parallelogram $ABA'B'$ formed by the two translates AB, $A'B'$ of a side in the support of μ. Replace this parallelogram with a line segment congruent to AA', and assign to this segment a density equal to the length of AB. Perform this operation for all dark gray parallelograms, and discard the white pieces of the puzzle. We obtain this way the restriction to $\Delta_{\omega(\mu)}$ of a measure $\mu^* \in \mathcal{M}^*_{\omega(\mu)}$, called the *dual* of μ. For the particular measure considered above, the support of the dual measure is pictured below. The densities are again equal on all edges in the support.



The location of the exit points of μ^* can be identified by noting that the distance between consecutive exit points (on the same side) is equal to the an exit density of μ, and the corresponding exit density is equal to the distance between two consecutive exit points of μ. The following picture illustrates this. The thicker arrows represent higher densities. The measure μ has weight $\omega(\mu) = 4$.



The rigidity of a measure is equivalent to a geometric property of its puzzle. Orient all the edges of the dark gray parallelograms in the puzzle of μ so that they point away from the acute angles. As shown in [5], μ is rigid if and only if the resulting oriented graph has no *gentle* cycle, i.e. a cycle with no sharp turns. The measure inflated above is not rigid, as demonstrated by the gentle cycle shown below.



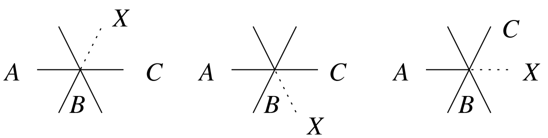
This characterization of rigidity easily implies that the dual of a rigid measure is also rigid. Indeed, μ and μ^* have the same puzzle.

3. Rigidity, descendance, and homology

Consider a measure $\mu \in \mathcal{M}_r$ for some $r > 0$, and let AB and BC be two segments in the support of μ containing no branch points in their interior. We will write $AB \to_\mu BC$ if one of the following two situations occurs:

(1) A, B, C are collinear, and there is a segment BX with $\sphericalangle XBC = 60°$ such that $\mu(BX) = 0$;
(2) $\sphericalangle ABC = 120°$, and there is a segment BX collinear with AB such that $\mu(BX) = 0$.

The figure below illustrates the relation '$\to_\mu$'; the dotted segments are assumed to have measure equal to zero.



The relation $AB \to_\mu BC$ always implies that the density of μ on BC is greater than or equal to the density on AB. More generally, if $A_0A_1, A_1A_2, \dots, A_{n-1}A_n$ are segments in the support of μ containing no branch points in their interior, we write $A_0A_1 \Rightarrow_\mu A_{n-1}A_n$ if $A_{i-1}A_i \to_\mu A_iA_{i+1}$ for $i = 1, 2, \dots, n$. In this case, the segment $A_{n-1}A_n$ is called a *descendant* of A_0A_1, and the sequence $A_0A_1 \cdots A_n$ is called a *descendance path*. The equivalence relation $AB \Leftrightarrow_\mu A'B'$ is defined by $AB \Rightarrow_\mu A'B'$ and $A'B' \Rightarrow_\mu AB$; in order to obtain an equivalence relation, we also allow $AB \Leftrightarrow_\mu AB$. A segment AB in the support of μ is called a *root edge* if the relation $A'B' \Rightarrow_\mu AB$ implies $AB \Rightarrow_\mu A'B'$. The following facts were proved in [1, Section 3] in the case that μ is a rigid measure.

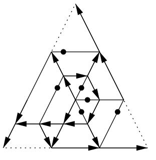
(1) If XY is in the support of μ and it is not a root edge, then there is at least one descendance path from a root edge to XY. Moreover, all descendance paths give XY the same orientation.
(2) For each root edge AB, there exists a measure $m \in \mathcal{M}_r$ supported by the descendants of AB, with density one on AB. This measure m is extremal, and it assigns integer densities to all edges.
(3) Let $m_1, m_2, \dots, m_k$ be the measures associated as in (2) to a maximal family of inequivalent root edges, and let δ_j the density of μ on the root edges of m_j. Then we have

$$\mu = \sum_{j=1}^{k} \delta_j m_j.$$

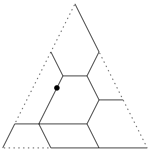
Thus the number of extremal summands of μ can be determined entirely from the geometry of the support of μ. In particular, any measure which has the same support as μ is rigid, and it can be written as

$$\sum_{j=1}^{k} \gamma_j m_j$$

for some positive constants γ_j. We illustrate this process with the following figure which represents the support of a rigid measure μ for which there are $k = 6$ extremal summands. Six root edges are marked with a dot, and all the other edges are oriented by descendance from one (or several) of these root edges. The supports of the six summands are easily determined.



We show one of these supports below. The other five are much simpler.



Two extremal measures will be said to be *inequivalent* if neither of them is a multiple of the other; this is equivalent to saying that the measures have different supports. The measures $m_1, m_2, \dots, m_k$ are mutually inequivalent since the root edge of m_j is not in the support of m_i for $i \neq j$.

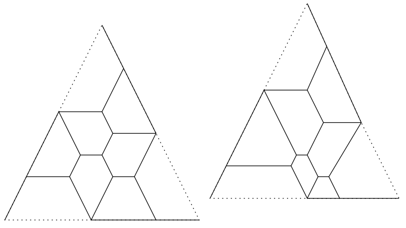
Proposition 3.1. *Let $\mu \in \mathcal{M}_r$ be a rigid measure, and let $A_1, A_2, \dots, A_q$ be its attachment points. Write μ as a sum of mutually inequivalent extremal measures $\mu = \sum_{i=1}^{k} \mu_i$, and denote by $\alpha_i^{(j)}$ the exit density of μ_j at the point A_i. Then the vectors $\alpha^{(j)} = (\alpha_i^{(j)})_{i=1}^{q} \in \mathbb{R}^q$ are linearly independent.*

Proof. Assume to the contrary that the vectors $\alpha^{(j)}$ are linearly dependent. After a permutation of the measures, we may assume that there exist an integer $q_0 \in \{1, 2, \dots, q\}$, and nonnegative numbers β_j such that $\beta_1 \neq 0$ and the measures $\sum_{i=1}^{q_0} \beta_i \mu_i$, $\sum_{i=q_0+1}^{q} \beta_i \mu_i$ have the same exit densities at all attachment points. Relation (2.1) implies that the exit densities are the same at all exit points. The supports of these measures are contained in the support of μ, hence they are both rigid. We deduce that

$$\sum_{i=1}^{q_0} \beta_i \mu_i = \sum_{i=q_0+1}^{q} \beta_i \mu_i,$$

and this is a contradiction because the measure on the right-hand side of this equation assigns zero density to some root edge of μ_1, unlike the left-hand side. $\square$

Consider now two measures $\mu \in \mathcal{M}_r$, $\mu' \in \mathcal{M}_{r'}$, and denote by $\mathcal{V}$ the collection of all vertices of white puzzle pieces in Δ_r determined by the support of μ. In other words, $\mathcal{V}$ consists of the branch points and the exit points of μ, plus the corners of Δ_r which are not exit points. Denote by $\mathcal{V}'$ the corresponding collection for μ'. We say that μ and μ' are *homologous* if there exists a bijection $\varphi : \mathcal{V} \to \mathcal{V}'$ such that for any two points $X, Y \in \mathcal{V}$ we have (a) the segment XY is an edge of a white piece if and only if $\varphi(X)\varphi(Y)$ is an edge of a white piece, and (b) if XY is an edge of a white piece, then $\varphi(X)\varphi(Y)$ is parallel to XY, and $\mu(XY)=0$ if and only if $\mu'(\varphi(X)\varphi(Y))=0$. The following figure shows the supports of two homologous measures.



The characterization of rigidity in terms of gentle cycles makes it obvious that if μ is rigid, and μ' is homologous to μ, then μ' is rigid as well. The two measures in the illustration above are rigid.

We say that μ' is *strictly homologous* to μ if μ' is homologous to μ, and each edge XY in the support of μ has the same density as its homologous edge $\varphi(X)\varphi(Y)$ in the support of μ'. Our main result follows from a careful analysis of the set

$$\mathcal{H}_\mu=\{\mu':\ \mu' \text{ is strictly homologous to } \mu\}.$$

To begin with, given $\mu'\in\mathcal{H}_\mu$, we have $\omega(\mu')=\omega(\mu)$, and μ'^* has precisely the same support as μ^*. Indeed, the lengths of the sides of the puzzle pieces of μ^* are precisely the densities of μ. Conversely, a measure $\mu'\in\mathcal{M}_{r'}$ such that $\omega(\mu')=\omega(\mu)$ and $\operatorname{supp}(\mu'^*)=\operatorname{supp}(\mu^*)$ necessarily belongs to $\mathcal{H}_\mu$.

Now, if μ is rigid then so is μ^*, and therefore the measures $\nu\in\mathcal{M}^*_{\omega(\mu)}$ with the same support as μ^* are given by the general formula

$$\nu=\sum_{j=1}^{p}\gamma_j\nu_j,$$

where $p=\operatorname{ext}(\mu^*)$, $\nu_1,\nu_2,\dots,\nu_p$ are inequivalent extremal measures, and $\gamma_j>0$ for all j. Thus there is a bijection between $\mathcal{H}_\mu$ and $\mathbb{R}_+^{\operatorname{ext}(\mu^*)}$. Given $\gamma\in\mathbb{R}_+^{\operatorname{ext}(\mu^*)}$, we set

$$r(\gamma)=\omega\left(\sum_{j=1}^{p}\gamma_j\nu_j\right),\quad \gamma=(\gamma_1,\gamma_2,\dots,\gamma_p),$$

and denote by $m(\gamma)\in\mathcal{M}_{r(\gamma)}$ the measure satisfying

$$m(\gamma)^*=\sum_{j=1}^{p}\gamma_j\nu_j.$$

Denote by $\mathcal{V}_\gamma$ the collection of all vertices of white puzzle in $\Delta_{r(\gamma)}$ determined by the support of $m(\gamma)$, and let $\varphi_\gamma:\mathcal{V}\to\mathcal{V}_\gamma$ be the bijection yielding the homology of μ and $m(\gamma)$. If $A_1,A_2,\dots,A_q$ are the attachment points of μ, then $\varphi_\gamma(A_1),\dots,\varphi_\gamma(A_q)$ are the attachment points of $m(\gamma)$. If $X_j(\gamma)=\varphi_\gamma(X_j)$, $j=1,2,3$, are the vertices of $\Delta_{r(\gamma)}$, we have $\varphi_\gamma(A_i)=X_j(\gamma)+\Phi_i(\gamma)w_j$ where $\Phi_i(\gamma)>0$ is the coordinate of $\varphi_\gamma(A_i)$. We also set $\Phi_0(\gamma)=r(\gamma)$. Theorem 1.1 follows from the following result.

Theorem 3.2. *Let μ be a rigid measure, and let the maps $\Phi_0,\Phi_1,\dots,\Phi_q$ be as defined above, with $q=\operatorname{att}(\mu)$.*

(1) *The map $\Phi=(\Phi_0,\Phi_1,\dots,\Phi_q):\mathbb{R}_+^{\operatorname{ext}(\mu^*)}\to\mathbb{R}^{\operatorname{att}(\mu)+1}$ is linear.*
(2) *The map Φ is one-to-one.*
(3) *The range of Φ has dimension* $\operatorname{att}(\mu)+1-\operatorname{ext}(\mu)$.

The linearity of Φ is easily verified. Indeed, the lengths of the edges of white pieces in the puzzle of $m(\gamma)$ are equal to the densities of the dual edges in the support of $\sum_{j=1}^{p}\gamma_j\nu_j$, and these are obviously linear functions of γ. Part (2) follows immediately from the rigidity of μ. Indeed, $\Phi(\gamma)=\Phi(\gamma')$ implies that the measures $m(\gamma)$ and $m(\gamma')$ have the same attachment points and the same exit densities, hence they must coincide by rigidity. We conclude that $\sum_{j=1}^{p}\gamma_j\nu_j=\sum_{j=1}^{p}\gamma'_j\nu_j$, so that $\gamma=\gamma'$ because the measures ν_j are linearly independent. Assertion (3) is not as obvious, but one inequality follows from the following result.

Lemma 3.3. *The range of Φ is contained in a subspace $\mathbb{V}\subset\mathbb{R}^{\mathrm{att}(\mu)+1}$ of codimension* $\mathrm{ext}(\mu)$.

Proof. Write $\mu=\mu_1+\mu_2+\cdots+\mu_k$ with $k=\mathrm{ext}(\mu)$, and the μ_j are mutually inequivalent extremal measures. Denote by $\alpha_i^{(j)}$ the exit density of μ_j at A_i. The trace identity (2.2) can be written as

$$\sum_{i=1}^{q}\alpha_i^{(j)}\Phi_i(\gamma)=\omega(\mu_j)\Phi_0(\gamma),\quad \gamma\in\mathbb{R}^p,$$

for $j=1,2,\dots,k$. According to Proposition 3.1, these equations are linearly independent, so they define a linear subspace of codimension $k=\mathrm{ext}(\mu)$. □

4. Immersions of trees

Extremal rigid measures have an underlying tree structure, first described in [2]. We will consider binary trees with a finite number of branch points (i.e., vertices of order three) and with no vertices of order one. In other words, the leaves of the tree do not have endpoints. Each edge between two branch points, and each leaf, will be assigned a number in $\{1,2,3\}$ in such a way that the three numbers assigned around each branch point are distinct. This number will be called the *type* of the edge. We will assume that each tree has at least one branch point. Given a tree T, an *immersion* of T is a continuous map $f:T\to\mathbb{R}^2$ satisfying the following conditions:

(1) each edge of type j joining two branch points is mapped homeomorphically onto a segment parallel to w_j,
(2) each leaf of type j is mapped homeomorphically onto a half-line parallel to w_j.

We note that the type issue can be avoided by considering only trees embedded in $\mathbb{R}^2$ and only orientation-preserving immersions.

Given an immersion f of a tree T, there is a measure $\mu_f\in\mathcal{M}$ supported by $f(T)$ such that the density of a segment in $f(T)$ equals the number of preimages of that segment under f. Measures of the form μ_f are called *tree measures.* Two immersions $f,g:T\to\mathbb{R}^2$ yield equal tree measures provided that $f(t)=g(t)$ for every branch point $t\in T$. In fact, even less information suffices to determine μ_f.

Proposition 4.1. *Choose a point $t_\ell\in\ell$ for each leaf ℓ of a tree T. Two immersions $f,g:T\to\mathbb{R}^2$ yield equal measures if $f(t_\ell)=g(t_\ell)$ for every ℓ.*

Proof. As observed above, it suffices to show that the value of f at each branch point is determined by the values $f(t_\ell)$. There must exist leaves ℓ_1 and ℓ_2 which meet at a branch point $t_0\in T$, and $f(t_0)$ is then precisely the intersection of the lines containing $f(\ell_1)$ and $f(\ell_2)$. These lines are determined by $f(t_{\ell_1})$ and $f(t_{\ell_2})$ because their directions are dictated by the types of the two leaves, and these types are distinct. Replace now the leaves ℓ_1 and ℓ_2 with a single leaf ℓ attached at t_0, and with type different from those of ℓ_1 and ℓ_2. Also define $t_\ell=t_0$. We obtain a new tree T' with one fewer leaves than T. Define an immersion f' of T' which agrees with f on the common part of T and T'. This operation reduces the proof of the proposition from T to T', and therefore we can proceed by induction from the trivial case of a tree with three leaves. □

The case of the tree with three leaves shows that the points $f(t_\ell)$ in the above proposition must satisfy a linear equation. In the case of tree measures in $\mathcal{M}_r$, this is essentially the trace identity. The following result shows that, other than this one equation, the points $f(t_\ell)$ can be perturbed more or less arbitrarily.

Proposition 4.2. *Let T be a tree, and $S \subset T$ a subset such that*

(1) *for each leaf ℓ of T, the intersection $\ell \cap S$ consists of a single point t_ℓ; and*
(2) *every point $s \in S$ is of the form t_ℓ for some leaf ℓ.*

Fix an immersion $f : T \to \mathbb{R}^2$ and a point $s_0 \in S$. For every $\varepsilon > 0$ there exists $\delta > 0$ with the following property: for any function $g_0 : S \setminus \{s_0\} \to \mathbb{R}^2$ such that

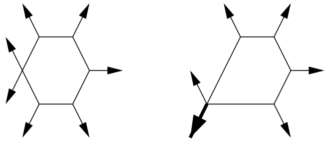
$$\left|g_0(s) - f(s)\right| < \delta, \quad s \in S \setminus \{s_0\},$$

there exists an immersion g of T such that $|f(t) - g(t)| < \varepsilon$ for all $t \in T$, and $g(s) = g_0(s)$ for $s \in S \setminus \{s_0\}$.

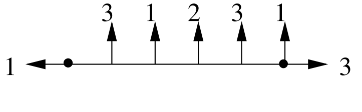
Proof. As in the preceding proposition, we proceed by induction on the number of leaves. If T has three leaves, the set S contains either one or three elements. In the case of one element, we can choose $\delta > 0$ arbitrarily and define $g = f$. In the case of three elements, choose leaves ℓ_1, ℓ_2 such that $t_{\ell_1} \neq s_0 \neq t_{\ell_2}$. These leaves will meet at the unique branch point t_0 of T, and the half-lines $f(\ell_1)$ and $f(\ell_2)$ meet at $f(t_0)$. The lines parallel to $f(\ell_1)$ and $f(\ell_2)$ and passing through $g_0(t_{\ell_1})$ and $g_0(t_{\ell_2})$, respectively, meet at a point A such that $|A - f(t_0)| < 2\delta$. The existence of g so that $g(t_0) = A$, $g(t_{\ell_1}) = g_0(t_{\ell_1})$, and $g(t_{\ell_0}) = g_0(t_{\ell_0})$ follows immediately provided that δ is sufficiently small. Assume now that T has more than 3 leaves, and the proposition has been proved for trees with fewer leaves. Choose two leaves ℓ_1, ℓ_2 such that $t_{\ell_1} \neq s_0 \neq t_{\ell_2}$ which intersect at t_0, and form a tree T' and an immersion f' as in the preceding proof. For the tree T' we choose the set $S' = (S \setminus \{t_{\ell_1}, t_{\ell_2}\}) \cup \{t_0\}$, and observe that $t_\ell = t_0$ for the new leaf ℓ. Define the map $g_0' : S' \setminus \{s_0\} \to \mathbb{R}^2$ by setting $g_0'(s) = g_0(s)$ for $s \in S \setminus \{t_{\ell_1}, t_{\ell_2}\}$, while $g'(t_0)$ is the intersection point of the lines parallel to $f(\ell_1)$ and $f(\ell_2)$ and passing through $g_0(t_{\ell_1})$ and $g_0(t_{\ell_2})$, respectively. Observe that $|g_0'(t_0) - f'(t_0)| < 2\delta$. The inductive hypothesis provides a positive number δ' corresponding to the immersion f'. Choosing $\delta' \leqslant \delta/2$, we deduce the existence of an immersion g' of T' such that $g'(s) = g_0(s)$ for $s \in S \setminus \{s_0, t_{\ell_1}, t_{\ell_2}\}$, $g'(t_0) = g_0'(t_0)$, and $|g'(t) - f'(t)| < \varepsilon$ for $t \in T'$. The existence of the required g follows now easily if δ is sufficiently small. Namely, define $g = g'$ on the common part of T and T', and extend this function appropriately to the leaves ℓ_1 and ℓ_2. □

The above proof yields $\delta \leqslant \varepsilon/2^b$, where b is the number of branch points of T, and $b + 2$ is the number of leaves. This is in fact the best estimate for small ε. It is easily seen from the preceding two proofs that the types of all the segments of a tree are determined by the types of the leaves. Even the type of a leaf is determined by the types of all the other leaves.

Different immersions of the same tree may yield measures which are not homologous. An example is pictured below, where the thicker line indicates density 2.



An immersion of the tree pictured below, taking equal values at the two points indicated by a dot, yields an immersion homologous to the second one in the preceding figure. (The numbers indicate the types of the leaves.)



We apply now the preceding results to a tree measure in $\mathcal{M}_r$.

Proposition 4.3. *Let $f: T \to \mathbb{R}^2$ be an immersion such that $\mu_f \in \mathcal{M}_r$ for some $r > 0$. We denote by $A_1, A_2, \dots, A_q$ the attachment points of μ_f, by $x_1, x_2, \dots, x_q \in (0, r]$ their coordinates, and by $\alpha_1, \alpha_2, \dots, \alpha_q$ the corresponding exit densities. For every $\varepsilon > 0$ there exists $\delta > 0$ with the following property: given $r' > 0$ and points $A'_1, A'_2, \dots, A'_q \in \partial\Delta_{r'}$ with coordinates $x'_1, \dots, x'_q \in (0, r']$ such that*

$$\sum_{j=1}^{q} \alpha_j x'_j = \omega(\mu_f) r'$$

and

$$\left| A'_j - A_j \right| < \delta, \quad j = 1, 2, \dots, q,$$

there exists an immersion $g: T \to \mathbb{R}^2$ such that $|f(t) - g(t)| < \varepsilon$ for all $t \in T$, μ_g belongs to $\mathcal{M}_{r'}$, it has exit points $A'_1, A'_2, \dots, A'_q$ and the corresponding exit densities are $\alpha_1, \alpha_2, \dots, \alpha_q$.

Proof. The case in which μ_f has only one exit point is trivial. Indeed, in that case the coordinate of the exit point is equal to r, and the trace identity implies that the point A' has coordinate r'. The map g can simply be constructed as the translate $g = f + A' - A$, and this will satisfy the requirements of the proposition if $\delta \leqslant \varepsilon$ is sufficiently small. We will therefore assume that μ_f has at least two exit points, in which case there exists $t_0 \in T$ such that $f(t_0) \in \Delta_r \setminus \partial\Delta_r$. For each simple path in T which starts at t_0 and ends with one of the leaves, there exists a first point s such that $f(s) \in \partial\Delta_r$. We denote by S the collection of all these points. We can write $S = \bigcup_{j=1}^{q} S_j$ so that $f(t) = A_j$ for $t \in S_j$. Moreover, the density α_j is precisely the cardinality of S_j. Fix an arbitrary point $s_0 \in S_1$, and let δ_0 be provided by Proposition 4.2, and choose $\delta < \delta_0$ such that 3δ is smaller than all the segments determined on $\partial\Delta_r$ by the points A_j and the corners of Δ_r. For each $s \in S \setminus \{s_0\}$ such that $f(s) = A_j$, we set $g_0(s) = A'_j$. The choice of δ implies the existence of an immersion $g: T \to \mathbb{R}^2$ such that $g(s) = g_0(s)$ for $s \in S \setminus \{s_0\}$. We can also assume that the point $A = g(s_0) \in \partial\Delta_{r'}$, and the shortest path from t_0 to s_0 contains no other points in $\partial\Delta_{r'}$. The choice of δ ensures that A is on the same side of $\Delta_{r'}$ as A'_1. Clearly the measure μ_g is in $\mathcal{M}_{r'}$, and its attachment points are A'_j, $j \geqslant 2$, with exit density α_j; A'_1 with exit density $\alpha_1 - 1$; and finally A with density 1. To conclude the proof, we will show that in fact $A = A'_1$. Denote indeed by x the coordinate of A, and write the trace identity for μ_g:

$$x + (\alpha_1 - 1)x'_1 + \sum_{j=2}^{q} \alpha_j x'_j = \omega(\mu_g) r'.$$

Since $\omega(\mu_g) = \omega(\mu_f)$, this equation, combined with the hypothesis, implies $x = x'_1$, and therefore $A = A'_1$, as claimed. $\square$

5. Perturbations of rigid measures

Fix a rigid measure μ, and assume that it assigns unit density to all of its root edges. As seen above, we can write μ as a sum of extremal measures

$$\mu = \sum_{j=1}^{k} \mu_j,$$

where $k = \operatorname{ext}(\mu)$, each μ_j assigns unit mass to its root edges, and the support of μ_j consists of all the descendants of some root edge e_j of μ. It was shown in [2] that μ_j is of the form μ_{f_j} for some immersion f_j of a tree T_j. More precisely, choose for each j a point P_j in the interior of e_j, and a point $t_j \in T_j$ such that $f(t_j) = P_j$. Then every simple path starting at t_j is mapped to a descendance path starting at P_j; in fact, every descendance path starting at P_j can be obtained this way, and this is essentially how the tree T_j is constructed. We denote by $A_1, A_2, \dots, A_q$ the attachment points of μ, where $q = \operatorname{att}(\mu)$, and we let $\alpha_i^{(j)}$ be the exit density of μ_j at the point A_i. Clearly, the range of the map Φ considered in Theorem 3.2 contains the point $(r, x_1, x_2, \dots, x_q)$, where x_i is the coordinate of the point A_i. The space $\mathbb{V}$ of Lemma 3.3 consists of those triples $(r', x'_1, x'_2, \dots, x'_q) \in \mathbb{R}^{\operatorname{att}(\mu)+1}$ satisfying the linearly independent equations

$$\sum_{i=1}^{q} \alpha_i^{(j)} x'_i = \omega(\mu_j) r', \quad j = 1, 2, \dots, k. \tag{5.1}$$

Therefore assertion (3) of Theorem 3.2 follows from Lemma 3.3 and the following result.

Proposition 5.1. *With the above notation, there exists $\delta > 0$ such that any point $(r', x'_1, x'_2, \dots, x'_q) \in \mathbb{R}^{\operatorname{att}(\mu)+1}$ satisfying $|r' - r| < \delta$, $|x'_i - x_i| < \delta$ for $i = 1, 2, \dots, q$, and Eqs. (5.1), belongs to the range of Φ.*

Proof. Denote, as before, by $\mathcal{V}$ the set consisting of all the vertices of the polygons into which $\operatorname{supp}(\mu)$ divides Δ_r, and denote by 5ε the shortest distance between two points in $\mathcal{V}$. Choose $\delta_0 < \varepsilon$ satisfying the conclusion of Proposition 4.3 for each of the immersions f_j, $j = 1, 2, \dots, k$. We will show that our proposition is satisfied for $\delta = \delta_0/2$. Assume indeed $(r', x'_1, x'_2, \dots, x'_q) \in \mathbb{R}^{\operatorname{att}(\mu)+1}$ satisfies the hypothesis, and denote by $A'_i \in \partial\Delta_r$ the point with coordinate x'_i such that $|A'_i - A_i| < \delta$. Proposition 4.3 implies the existence of immersions g_j of T_j, $j = 1, 2, \dots, k$, such that the measures μ_{g_j} belong to $\mathcal{M}_{r'}$, $|g_j(t) - f_j(t)| < \varepsilon$ for $t \in T_j$, and μ_{g_j} has exit density $\alpha_i^{(j)}$ at the point A'_i. Finally, set $\mu' = \sum_{j=1}^{k} \mu_{g_j}$. The following picture illustrates a typical branch point of μ, along with its hypothetical perturbation in the support of μ'.

To conclude the proof, it will suffice to show that μ' is strictly homologous to μ. Denote by $\mathcal{V}'$ the collection of vertices of white puzzle pieces corresponding to μ', and define a map $\psi : \mathcal{V}' \to \mathcal{V}$ which associates to each point $B' \in \mathcal{V}'$ the *closest* point $B \in \mathcal{V}$. Observe that the points $B' \in \mathcal{V}'$ not on the boundary of $\Delta_{r'}$ are of two kinds. The first are of the form $g_j(y)$, with y a branch point of the tree T_j, and in this case we have $\psi(B') = f_j(y)$. The second kind arise as intersections $g_i(e_i) \cap g_j(e_j)$, where e_i and e_j are edges of T_i and T_j respectively, and their types are different. In this case we have $\psi(B') = f_i(e_i) \cap f_j(e_j)$. In both cases, the distance from B' to $\psi(B')$ is less than 2ε. Hence our choice of ε implies that this map is well defined.

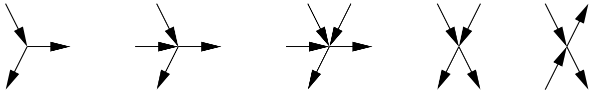
Clearly we have $\psi(A'_i) = A_i$ and $\psi^{-1}(A_i) = \{A'_i\}$ for every i. More generally, for every point $B \in \mathcal{V}$, let us denote by $\kappa(B)$ the cardinality of $\psi^{-1}(B)$, and observe that $\kappa(B) > 0$ for every $B \in \mathcal{V}$. In other words, ψ is onto. (For the above picture, we would have $\kappa = 6$.) Assume that BC is a white piece edge in the support of μ, and $\mu(BC) = M$. In other words, there exist $j_1, j_2, \dots, j_M \in \{1, 2, \dots, k\}$ and edges e_{j_i} in T_{j_i} such that $f_{j_i}(e_{j_i})$ contains BC for $i = 1, 2, \dots, M$. Thus there are segments $e'_{j_i} \subset e_{j_i}$ such that $f_{j_i}(e'_{j_i}) = BC$. The endpoints of the segments $g_{j_i}(e'_{j_i})$ are within 2ε from some points $B'_i, C'_i \in \mathcal{V}'$, and in fact $\{B'_1, B'_2, \dots, B'_M\} \in \psi^{-1}(B)$ and $\{C'_1, C'_2, \dots, C'_M\} \in \psi^{-1}(C)$. The proposition will therefore be proved if we can show that $\kappa(B) = 1$ for every $B \in \mathcal{V}$. Indeed, if that were the case we would have $\mu'(B'_1 C'_1) = \mu(BC) = M$ because all the edges of the trees T_j, other than $e_{j_1}, \dots, e_{j_M}$, are mapped by g_j to segments which do not contain $B'_1 C'_1$. The map ψ would thus be the bijection witnessing the strict homology of μ and μ'.

We have already noted that $\kappa(B) = 1$ if $B \in \partial\Delta_r$. Select for each $j \in \{1, 2, \dots, k\}$ a root edge e_j of μ which is also a root edge for μ_j, pick a point P_j in the interior of e_j, and let $t_j \in T_j$ satisfy $f(t_j) = P_j$. Given the points $A, B \in \mathcal{V}$, we will write $A \to B$ if there is a descendance path $Y_0 Y_1 \cdots Y_n$ such that $Y_{n-1} = A$, $Y_n = P$, and $Y_0 = P_j$ for some $j \in \{1, 2, \dots, k\}$. Note that we cannot have $A \to B$

and $B \to A$. Define now sets $\mathcal{V}_0 \subset \mathcal{V}_1 \subset \cdots \subset \mathcal{V}$ as follows: $\mathcal{V}_0 = \mathcal{V} \cap \partial\Delta_r$, and inductively $\mathcal{V}_{n+1}$ consists of those vertices B with the property that

$$\{C \in \mathcal{V}: B \to C\} \subset \mathcal{V}_n.$$

The properties of descendance paths for a rigid measure ensure that we have $\mathcal{V}_n = \mathcal{V}$ for sufficiently large n. We proceed to prove by induction on n that $\kappa(B) = 1$ for every $B \in \mathcal{V}_n$. We already know that this is true for $n = 0$, so assume that it has been proved for all $n < N$, and let $B \in \mathcal{V}_N$. Since $N > 0$, B is in the interior of Δ_r, and B is a branch point of μ. The support of μ in the neighborhood of μ must be in one of the following situations up to a rotation or reflection. (This is easily deduced from the fact that the edges in the support of μ are given a unique orientation by the relation of descendance from any root edge. Thus, for instance, the support of μ cannot contain six edges meeting at B since this would not allow descendance past that point.)

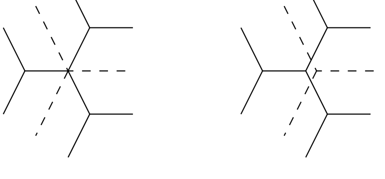


The arrows indicate the orientation given by descendance from one of the points P_j. In all of these situations, there are precisely two points $C_1, C_2 \in \mathcal{V}_{N-1}$ such that $B \to C_1$ and $B \to C_2$. Moreover, B is the intersection of the lines passing through C_1, C_2 and parallel to w_{ℓ_1}, w_{ℓ_2}, respectively, where $\ell_1, \ell_2 \in \{1, 2, 3\}$ are distinct. The inductive hypothesis implies the existence of unique points $C_1' \in \psi^{-1}(C_1)$ and $C_2' \in \psi^{-1}(C_2)$. Denote by X the intersection of the lines passing through C_1', C_2' and parallel to w_{ℓ_1}, w_{ℓ_2}, respectively. We will conclude the proof by showing that $\psi^{-1}(B) = \{X\}$. Assume indeed that $B' \in \psi^{-1}(B)$. To do this, it suffices to show that, given j and a point $s_j \in T_j$ such that $f_j(s_j) = B$, the map g_j maps some point in the neighborhood of s_j to X. There are two situations to consider.

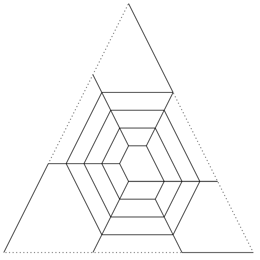
(1) If s_j is a branch point for T_j, then two of the branches are mapped by f_j to line segments passing through C_1, C_2 and parallel to w_{ℓ_1}, w_{ℓ_2}, respectively. These two branches are mapped by g_j to segments through C_1', C_2' and parallel to w_{ℓ_1}, w_{ℓ_2}, respectively, and thus $g(s_j) = X$ in this case.
(2) If s_j is not a branch point for T_j, then f_j maps the branch containing s_j to a line passing through C_i and parallel to w_{ℓ_i} for $i = 1$ or $i = 2$. It follows that g_j maps this branch to a line passing through C_i' and parallel to w_{ℓ_i}, and this line passes through X. It follows that $g_j(s_j') = X$ for some s_j' on this branch.

The proposition follows. $\square$

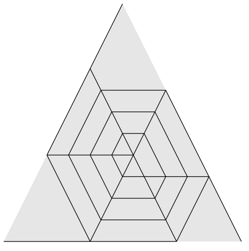
The rigidity assumption cannot be discarded from the hypothesis of the preceding proposition, even for measures which are sums of rigid tree measures. The following figure shows a sum of two tree measures (the support of one of them in dashed lines) which is perturbed to a measure which is not homologous to the original one.



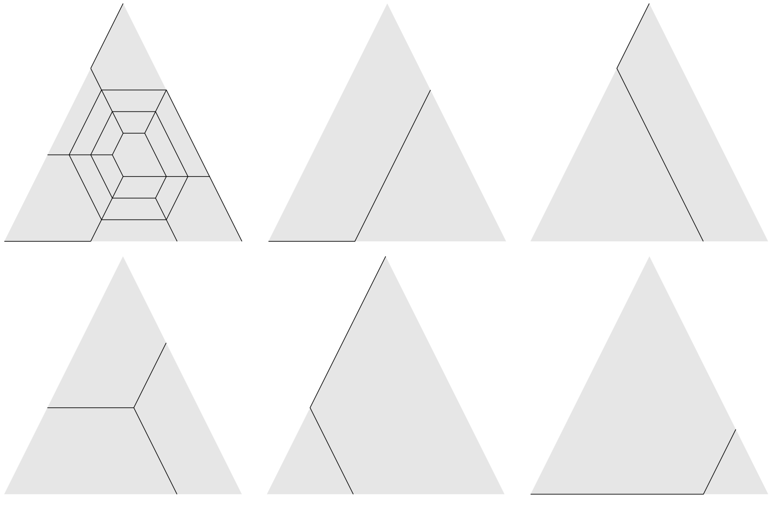
We conclude with an illustration of our main result. The following figure represents the support (intersected with Δ_{14}) of a rigid, extremal tree measure $\mu \in \mathcal{M}_{14}$. Clearly we have $\operatorname{att}(\mu) = 6$ and, if we assign unit density to its root edges (which include, for instance, the edges of the central pentagon) we have $\omega(\mu) = 11$.



The next figure represents $\mathrm{supp}(\mu^*) \cap \Delta_{11}$.



The measure μ^* is the sum of six extremal measures, as implied by Theorem 1.1. The supports of these measures are depicted below.



The first of these measures has weight 9 and the remaining five have weight 1. The reader familiar with the results of [1] will be able to verify that, among these six measures, the first measure is the only minimal one relative to precedence.

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